ON PRODUCT SUMMABILITY OF FOURIER SERIES USING MATRIX EULER METHOD

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ABSTRACT
In this paper, a theorem on product summability of Fourier series using Matrix-Euler method is proved.

KEYWORDS: A - mean, A(\(E, z\)) - product mean, Fourier series.

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I. INTRODUCTION

Let \(\sum a_n\) be a given infinite series with the sequence of partial sums \(\{s_n\}\). Let \(A = (a_{mn})_{m,n=0}^{\infty}\) be a matrix .Then the sequence –to-sequence transformation

\[
t_m = \sum_{\nu=0}^{m} a_{m\nu} s_{\nu}, m = 1,2,\cdots
\]

defines the sequence \(\{t_m\}\) of the \(A\) -mean of the sequence \(\{s_n\}\). If

\[
t_m \to s, \text{ as } m \to \infty,
\]

then the series \(\sum a_n\) is said to be \(A\) summable to \(s\).

The conditions for regularity of \(A\) -summability are easily seen to be[3]

(i) \(\sup_m \sum_{n=0}^{\infty} |a_{mn}|<H\) where \(H\) is an absolute constant.

(ii) \(\lim_{m \to \infty} a_{mn} = 0\)

(iii) \(\lim_{m \to \infty} \sum_{n=0}^{\infty} a_{mn} = 1\)

If

\[
(E, z) = E_nz = \frac{1}{(1+z)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} z^{n-\nu} s_{\nu} \to s, \text{ as } n \to \infty,
\]

then the series \(\sum a_n\) is said to be summable \((E, z)\) to a definite number \(s\).

Let
\[ T_n = \sum_{k=0}^{n} \frac{a_{nk}}{(1 + z)^{k}} \sum_{\nu=0}^{k} \left( \begin{array}{c} k \\ \nu \end{array} \right) z^{k-\nu} s_{\nu} \rightarrow s \quad \text{as} \quad n \rightarrow \infty, \]

then the series \( \sum a_{n} \) is said to be summable to \( s \) by the \( A(E, z) \) method.

It is known [1] that \( (E, z) \) is regular. It is supposed that the method \( A(E, z) \) is regular throughout this paper.

Let \( f(t) \) be a periodic function with period \( 2\pi \), integrable in the sense of Lebesgue over \((-\pi, \pi)\) then

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \]

is the Fourier series associated with \( f \).

We use the following notation throughout this paper

\[
\Phi(t) = f(x + t) + f(x - t) - 2f(x),
\]

\[
K_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{nk}}{(1 + z)^{k}} \sum_{\nu=0}^{k} \left( \begin{array}{c} k \\ \nu \end{array} \right) z^{k-\nu} \sin \left( \frac{\nu + 1}{2} t \right) \sin \frac{t}{2}.
\]

II. KNOWN THEOREM

Dealing with \((N, p_n)(E, z)\) method of a Fourier series, Nigam, et. al. [2] proved the following theorem:

**THEOREM-2.1:**

Let \( \{p_n\} \) be a positive, monotonic, non-increasing sequence of real constants such that

\[
P_n = \sum_{p=0}^{n} p_{\nu} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]

If

\[
\Phi(t) = \left[ \phi(u) \right] du = O\left( \frac{t}{\alpha (1 - \frac{1}{t})} \right), \quad \text{as} \quad t \rightarrow +0
\]

and

\[
\alpha(n) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty
\]

where \( \alpha(t) \) is a positive, non-increasing function of \( t \), then the Fourier series \( \sum_{n=0}^{\infty} A_n(t) \) is summable \((N, p_n)(E, z)\) to \( f(x) \) at the point \( t = x \).

In this paper, we have generalized it to \( A(E, z) \) summability of Fourier series.

III. MAIN THEOREM

**THEOREM-3.1:**

Let \( A = (a_{mn})_{m,n=0}^{\infty} \) be a regular matrix and
\( (3.1) \quad \Phi(t) = \int_0^t |\phi(u)| \, du = O\left(\frac{t}{\alpha(1/t)}\right), \text{ as } \ t \to +0 \)

where \( \alpha(t) \) is positive, non-increasing function of \( t \) and

\( (3.2) \quad \alpha(n) \to \infty \text{ as } n \to \infty, \)

then the Fourier series \( \sum_{n=0}^\infty A_n(t) \) is summable \( A(E, z) \) at the point \( t \).

IV. REQUIRED LEMMAS

We require the following Lemmas to prove the theorem.

**LEMMA 4.1:**

\[ |K_n(t)| = O(n) \quad 0 \leq t \leq \frac{1}{n+1}. \]

**PROOF:**

For \( 0 \leq t \leq \frac{1}{n+1} \), we have \( \sin nt \leq nsint \) then

\[ |K_n(t)| \leq \frac{1}{2\pi} \sum_{k=0}^n \sum_{v=0}^k \frac{a_{nk}}{(1+z)^k} \left(\frac{k}{v}\right)^{k-v} \frac{(2v+1)\sin \frac{t}{2}}{\sin \frac{t}{2}} \]

\[ \leq \frac{1}{2\pi} \sum_{k=0}^n \sum_{v=0}^k \frac{a_{nk}}{(1+z)^k} \left(\frac{k}{v}\right)^{k-v} \]

\[ = \frac{1}{2\pi} \sum_{k=0}^n \frac{a_{nk} (2k+1)}{(1+z)^k} \left(\frac{k}{v}\right)^{k-v} \]

\[ = \frac{2n+1}{2\pi} \sum_{k=0}^n a_{nk} \]

\[ = O(n). \]

**LEMMA 4.2:**

\[ |K_n(t)| = O\left(\frac{1}{t}\right) \quad \text{for } \frac{1}{n} \leq t \leq \pi. \]

**PROOF:**

For \( \frac{1}{n} \leq t \leq \pi \), we have by Jordan’s lemma, \( \sin \left(\frac{t}{2}\right) \geq \frac{t}{\pi} \), \( \sin nt \leq 1 \). Then

\[ |K_n(t)| \leq \frac{1}{2\pi} \sum_{k=0}^n \sum_{v=0}^k \frac{a_{nk}}{(1+z)^k} \left(\frac{k}{v}\right)^{k-v} \frac{\sin \left(v + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \]

\[ \leq \frac{1}{2\pi} \sum_{k=0}^n \frac{a_{nk}}{(1+z)^k} \sum_{v=0}^k \left(\frac{k}{v}\right)^{k-v} \left(\frac{\pi}{t}\right) \]
\[
\frac{1}{2t} \sum_{k=0}^{a} \frac{a_{nk}}{(1+z)^k} (1+z)^k.
\]

\[
= O\left( \frac{1}{t} \right).
\]

V. PROOF OF THE THEOREM 3.1

If \( s_n \) is the n-th partial sum of the Fourier series \( \sum_{n=0}^{\infty} A_n(t) \) of \( f(t) \), then by using Riemann-Lebesgue theorem, following Titchmarsh [4] we have

\[
s_n - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\sin \left( \frac{n+1}{2} t \right)}{\sin \left( \frac{t}{2} \right)} \sin \left( \frac{n+1}{2} t \right) dt.
\]

Thus, the \( E(z) \) transform \( E_n^z \) of \( s_n \) is given by

\[
E_n^z - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\phi(t)}{\sin \left( \frac{t}{2} \right)} \sum_{k=0}^{n} \frac{k}{v} z^{n-k} \sin \left( \frac{k+1}{t} \right) \sin \left( \frac{v+1}{t} \right) dt.
\]

If \( T_n \) denote the \( A(E, z) \) transform of \( s_n \), we then have

\[
T_n - f(x) = \frac{1}{2\pi} \sum_{k=0}^{a} \frac{a_{nk}}{(1+z)^k} \int_{0}^{\pi} \frac{\phi(t)}{\sin \left( \frac{t}{2} \right)} \sum_{v=0}^{k} \frac{k}{v} z^{k-v} \sin \left( \frac{v+1}{t} \right) dt.
\]

In order to prove the theorem, under an assumption, it is sufficient to show that

\[
\left| \int_{0}^{\pi} \phi(t) K_n(t) dt \right| = O(1) \quad \text{as} \quad n \to \infty
\]

For \( 0 < \delta < \pi \), we have \( T_n - f(x) = \int_{0}^{\pi} \phi(t) K_n(t) dt \)

\[
= \left( \int_{0}^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\delta} + \int_{\frac{\pi}{n}}^{\pi} \right) \phi(t) K_n(t) dt.
\]

\[
= I_1 + I_2 + I_3, \quad \text{say}
\]

Now

\[
|I_1| = \left| \int_{0}^{\frac{\pi}{n}} \phi(t) K_n(t) dt \right| \leq \int_{0}^{\frac{\pi}{n}} |\phi(t)| |K_n(t)| dt.
\]

\[
\leq O(n) \int_{0}^{\frac{\pi}{n}} |\phi(t)| dt, \quad \text{Using Lemma -1}
\]

\[
= O(n) \left\{ O\left( \frac{1}{n\alpha(n)} \right) \right\}, \quad \text{using (3.1)}.
\]
\[ = O \left( \frac{1}{\alpha(n)} \right), \quad \text{as} \quad n \to \infty. \]
\[ = O(1), \quad \text{as} \quad n \to \infty, \quad \text{using (3.2)}. \]

Next
\[
|I_2| \leq \left| \int_{1/n}^{\delta} \left| \phi(t) \right| K_n(t) \ dt \right|
\]
\[ = O \left( \int_{1/n}^{\delta} \left| \phi(t) \right| dt \right), \quad \text{using lemma -2} \]
\[ = O \left\{ \frac{\Phi(t)}{t} \right\} _{1/n}^{\delta} + \delta \frac{\Phi(t)}{t^2} dt \right\} .
\]
\[ = O \left\{ \int_{1/\delta}^{\delta} \frac{1}{t} \left( \frac{1}{\alpha(t)} \right) \left( \frac{1}{u\alpha(u)} \right) \right\}, \quad \text{where} \ u=1/t
\]
\[ \quad \text{and} \quad 0 < \delta < 1 . \]

(using second mean-value theorem for the integral in the 2nd term as \( \alpha(n) \) is monotonic)
\[ = O(1) + O(1), \quad \text{as} \quad n \to \infty, \quad \text{using (3.2)} \]
\[ = O(1), \quad \text{as} \quad n \to \infty. \]

Finally
\[
|I_3| \leq \int_{\delta}^{\pi} \left| \phi(t) \right| K_n(t) \ dt = O(1) \quad \text{as} \quad n \to \infty.
\]
by using \ Riemann –Lebesgue theorem and the regularity condition of the method of summability. 
Thus, 
\[ T_n - f(x) = O(1), \quad \text{as} \quad n \to \infty. \]
This completes the proof of the theorem.

VI. CONCLUSION

Thus, Product Summability of Fourier series by Matrix-Euler method generalizes the \((N, p_n)\theta(E, z)\) - Product Summability of Fourier series and \((N, p_n, q_n)\theta(E, z)\) - Product Summability of Fourier series.

REFERENCES

Authors Biography

B. P. Padhy has got his Ph.D from Berhampur University in 2012. He has 14 years of teaching experience. He has published around 15 research articles in various International and National journals of repute. He has also published a book Entitled “Summability Methods and Applications” under Lap Lambert Academic Publishing GmbH & Co.KG, Germany. Presently he is working as an Assistant Professor in mathematics in Roland Institute of Technology, Berhampur, Odisha.

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Mahendra Misra has got his Ph.D in the year 1997 from Berhampur University. He has published around 30 research articles in various International and National journals of repute. To his credit he has produced three Ph.Ds under his guidance. Presently he is working as H.O.D of P.G. Department of Mathematics in N.C. College (Autonomous), Jajpur, Odisha.